The combinatorial game theory of Reverse Hex

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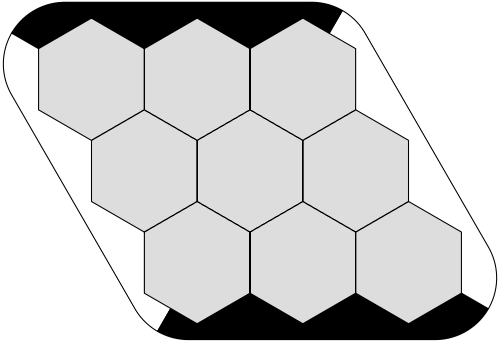
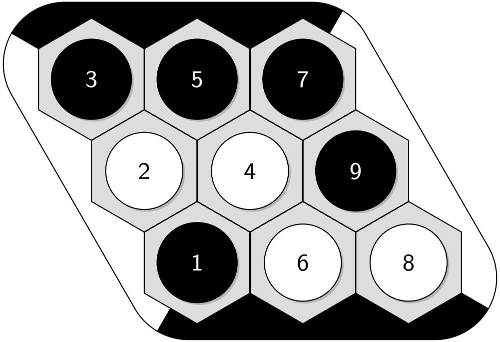
(For the PDF of these slides, visit "hockaday.neocities.org")

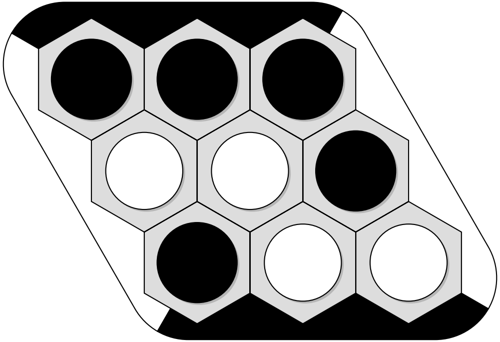
What is Reverse Hex (Rex)?



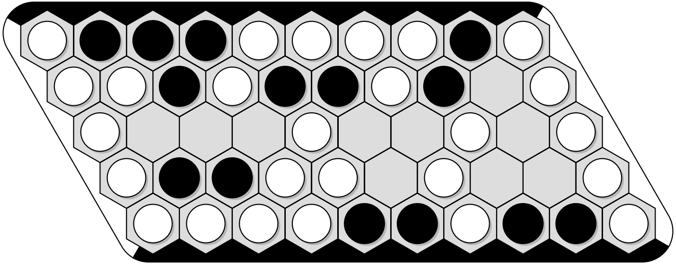
What is Reverse Hex (REX)?



What is Reverse Hex (REX)?

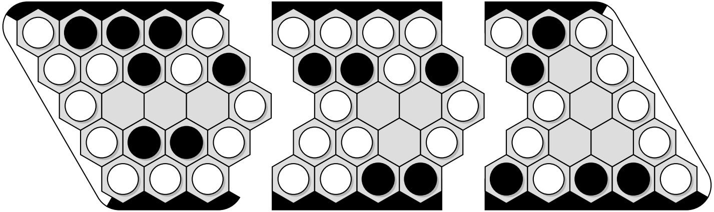


Example: a game of REX with 10 empty cells



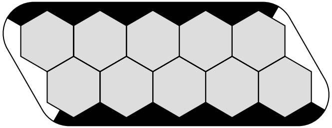
$$G = \{\, G^L \mid G^R \,\}$$

Representation as the sum of three smaller games

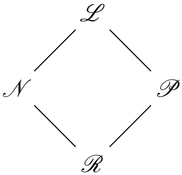


G_1 G_2 G_3

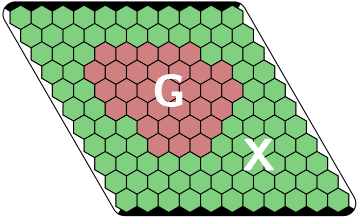
When do we have outcome classes in Rex



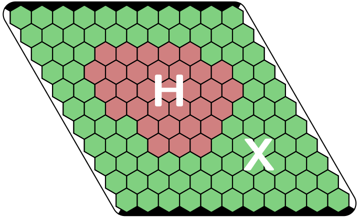
$$o(G) = \begin{cases} \mathscr{L} & 1^{st} \text{ and } 2^{nd} \text{ player win for Left} \\ \mathscr{N} & 1^{st} \text{ player win for both} \\ \mathscr{P} & 2^{nd} \text{ player win for both} \\ \mathscr{R} & 1^{st} \text{ and } 2^{nd} \text{ player win for Right} \end{cases}$$



Contextual (extrinsic) order



$\leq$



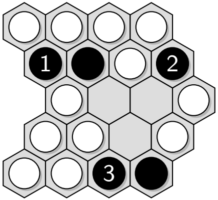
$G + X$ $\qquad$ $H + X$

$G \leq^c H$ if and only if, for all appropriate contexts X, $\quad o(G+X) \leq o(H+X)$

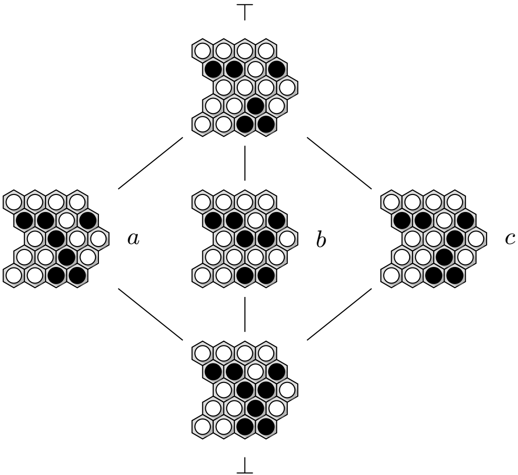
Special properties of REX

1. Antimonotonicity : $\bullet \leq \circ$.
2. $*$ antimonotonicity : $\bullet_{x}\langle * \rangle \leq \langle x \rangle \leq \circ_{x}\langle * \rangle$.
3. Look-ahead property: in an even position (a position with an even # of empty cells), a second player strategy can be translated into a first player strategy.

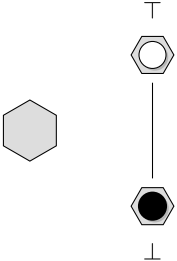
Atoms and posets: equivalence classes of Rex outcomes



G_2

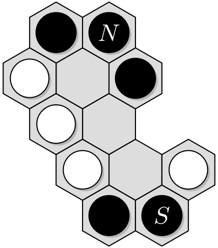


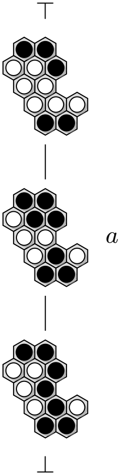
Examples: positions and their outcome posets



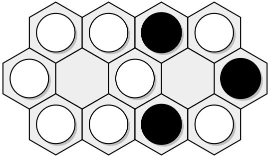
This is the two element poset, referred to as the boolean poset or $\mathbb{B}$.

Examples: positions and their outcome posets





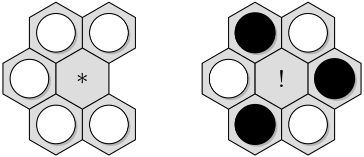
Properties of games: dead cells



A position is dead if its outcome poset is the one element poset. The cells in a dead position are called *dead cells*.

Which of these cells are dead?

Properties of games: dead cells



A position is dead if its outcome poset has only one element (denoted by 0). The cells in a dead position are called *dead cells*.

We represent a dead cell as $* = \{\, 0 \mid 0 \,\}$.

For any G, we have $G + [0] = G$.

We say G is $*$-*antimonotone* if:

$$\forall G^L \ \forall G^R : \ \ G^L + * \leq G \leq G^R + *.$$

As mentioned before, in $\textsc{Rex}$,

Parity:

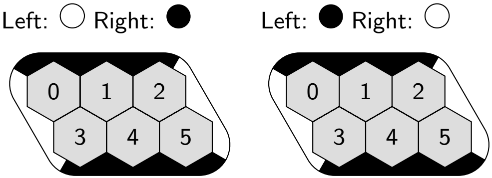
- G is *odd* $\iff$ all options of G are even and G is not atomic.
- G is *even* $\iff$ all options of G are odd.

If a game is $*$-antimonotone, then it has parity.

Look-ahead property:

- If G is odd, then having a 1^{st} player strategy implies having a 2^{nd} player strategy.
- If G is even, then having a 2^{nd} player strategy implies having a 1^{st} player strategy.

First player copycat strategy!



We say a game G over $\mathbb{B}$ is premotive $\iff$ Left has a 1^{st} player strategy in the game $G^{op} + G$ to win in G or G^{op}.

If a game is premotive, then it has the look-ahead property.

This order is similar to the order on normal play games when G and H are composite. $G \leq H :=$

- $\forall G^L$, $G^L \lhd H$, and
- $\forall H^R$, $G \lhd H^R$, and
- if G and H are atomic games g, h, then $g \leq h$.

$G \lhd H :=$

- $\exists G^R$ s.t. $G^R \leq H$ or
- $\exists H^L$ s.t. $G \leq H^L$.

We say that $G \simeq H$ when $G \leq H$ and $H \leq G$.

The intrinsic order is not inherently transitive

Here we show our favorite game:

$$F = \{\, \{\, \bot \mid \bot \,\} \mid \{\, \top \mid \top \,\} \,\}$$

$$\top \leq F \leq \bot.$$

The intrinsic order becomes transitive when restricted to premotive games.

In fact, the intrinsic order is equivalent to the contextual order when restricted to premotive games with parity.

The game F is the smallest game over $\mathbb{B}$ that has parity and is not premotive.

Reversibility:

Let $G = \{ H, G^L \mid G^R \}$ be a game such that $K \leq G$ for some right option K of H.

When we *bypass the reversible left option* from the game G it becomes the game $G' = \{ K^L, G^L \mid G^R \}$. Here, K^L denotes all left options of K when K is composite, and when k is atomic $K^L = \{ \bot \mid k \}$.

In Rex, $k \simeq \{ \{ \bot \mid k \} \mid \{ k \mid \top \} \}$.

Domination is as usual.

With both of these, we can find canonical forms of Rex positions!

Questions yet to be revealed:

- Which games are Rex-realizable?
- What is the computational complexity of Rex?
- Which properties from Hex carry over into Rex?

THANK YOU

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